May 26, 2016 Math 2X03 Dr. Prayat Poudel

Midterm A 150 Minutes

Full Name	Student I.D.
Solution	

THIS EXAMINATION PAPER INCLUDES **9** PAGES AND **8** QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

INSTRUCTIONS: No aids except the standard Casio fx991 calculator are permitted.

Problem	Points	Score
1	15	
2	10	
3	10	
4	15	
5	15	
6	15	
7	15	
8	5	
Total:	100	

1. (15 points) Find the mass of the solid tetrahedron with vertices $(0,0,0)$, $(\frac{1}{3},0,0)$, $(0,1,0)$, and $(0,0,1)$, and density function $\rho(x,y,z)=x$.

The equation of the solid tetrahedron with $(0,0,0)$, $(\frac{1}{3},0,0)$, $(0,1,0)$, and $(0,0,1)$ is given by

$$\begin{aligned} &\frac{x}{\frac{1}{3}}+\frac{y}{1}+\frac{z}{1}=1 \\ &3x+y+z=1 \end{aligned}$$

Then the mass of tetrahedron is given by

$$\begin{aligned} m &= \iiint_E \rho(x,y,z)\,dV \\ &= \int_0^{\frac{1}{3}}\int_0^{1-3x}\int_0^{1-3x-y} x\,dz\,dy\,dx = \int_0^{\frac{1}{3}}\int_0^{1-3x} xz\Big|_{z=0}^{z=1-3x-y} dy\,dx \\ &= \int_0^{\frac{1}{3}}\int_0^{1-3x} x(1-3x-y)\,dy\,dx = \int_0^{\frac{1}{3}}\int_0^{1-3x} x-3x^2-xy\,dy\,dx \\ &= \int_0^{\frac{1}{3}}\int_0^{1-3x} xz\Big|_{z=0}^{z=1-3x-y} dy\,dx = \int_0^{\frac{1}{3}}\int_0^{1-3x} x(1-3x-y)\,dy\,dx \\ &= \int_0^{\frac{1}{3}}\int_0^{1-3x} x-3x^2-xy\,dy\,dx = \int_0^{\frac{1}{3}} xy-3x^2y-x\frac{y^2}{2}\Big|_{y=0}^{y=1-3x} dx \\ &= \int_0^{\frac{1}{3}} x(1-3x)-3x^2(1-3x)-x\frac{(1-3x)^2}{2}\,dx \\ &= \int_0^{\frac{1}{3}} \frac{x}{2}-3x^2+\frac{9}{2}x^3\,dx \\ &= \left[\frac{x^2}{4}-x^3+\frac{9}{8}x^4\right]_0^{\frac{1}{3}} \\ &= \frac{1}{216} \end{aligned}$$

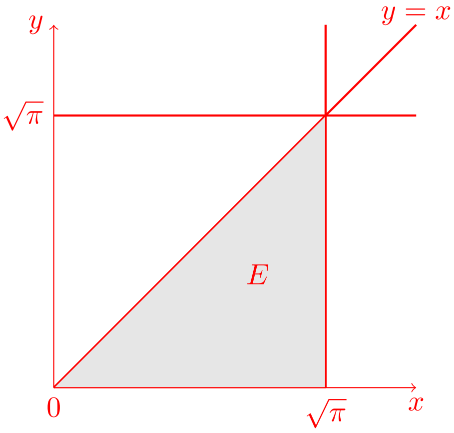
2. (10 points) Evaluate the integral

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2)\, dx\, dy$$

Let E denote the region of integration which is a Type II region,

$$E = \{(x, y) | 0 \le y \le \sqrt{\pi},\, y \le x \le \sqrt{\pi}\}$$

To solve the above integral we notice that the region of integration E is both a Type I and Type II. Therefore we will solve it be reversing the order of integration. The region of integration is the shaded region in the figure below.



Therefore, E can be expressed as a Type I region :

$$E = \{(x, y) | 0 \le x \le \sqrt{\pi},\, 0 \le y \le x\}$$

$$\begin{aligned} \int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2)\, dx\, dy &= \int_0^{\sqrt{\pi}} \int_0^{x} \cos(x^2)\, dy\, dx \\ &= \int_0^{\sqrt{\pi}} y \cos(x^2)\Big|_{y=0}^{y=x} dx = \int_0^{\sqrt{\pi}} x \cos(x^2)\, dx \end{aligned}$$

Use substitution $u = x^2$ and change limits of integration to u

$$\begin{aligned} &= \int_0^{\pi} \frac{1}{2} \cos u\, du \\ &= \left[\frac{1}{2} \sin u\right]_0^{\pi} = \frac{1}{2}(0 - 0) = 0 \end{aligned}$$

3. (10 points) The **Average value** of a function f of two variables defined on a region R is defined to be

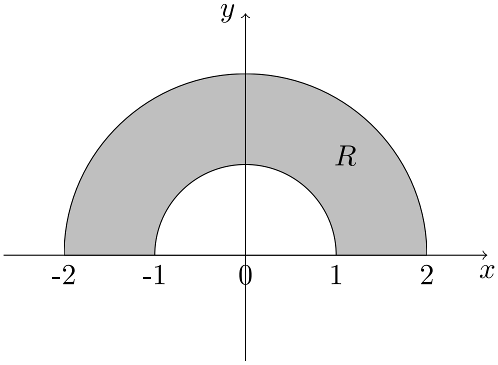
$$f_{ave} = \frac{1}{A(R)} \iint_R f(x,y)\,dA$$

where $A(R)$ denotes the area of the region R.

Ler R be the region bounded by a semi-circle of radius 2, a semi-circle of radius 1, and the x-axis (this R lies in the region $\{y > 0\}$). Compute the average value of the function

$$f(x,y) = e^{x^2-y^2}$$

over R.



The area of region R is

$$\begin{aligned} A(R) &= \frac{1}{2}\pi(2)^2 - \frac{1}{2}\pi(1)^2 \\ &= \frac{3}{2}\pi \end{aligned}$$

The region R in polar coordinates is given by

$$R = \{(r,\theta) | 1 \le r \le 2,\, 0 \le \theta \le \pi\}.$$

Then

$$\begin{aligned} f_{ave} &= \frac{1}{A(R)} \iint_R f(x,y)\,dA \\ &= \frac{1}{3\pi/2} \int_0^{\pi} \int_1^2 e^{r^2} r\,dr\,d\theta \\ &= \frac{2}{3\pi} \int_0^{\pi} d\theta \int_1^2 e^{r^2} r\,dr \quad \text{[Use substitution } u = r^2\text{]} \\ &= \frac{2}{3\pi}\pi \left[\frac{1}{2} e^{r^2}\right]_{r=1}^{r=2} \\ &= \frac{1}{3}\left(e^4 - e^1\right) \end{aligned}$$

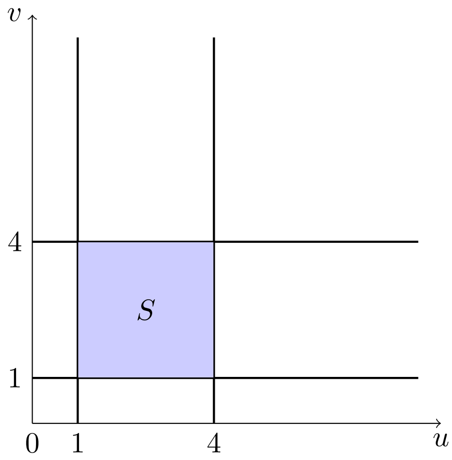
4. (15 points) Use the transformation $T(u,v) = \langle u, u+v \rangle$ (i.e $x(u,v) = u$, $y(u,v) = u+v$) to evaluate the integral,

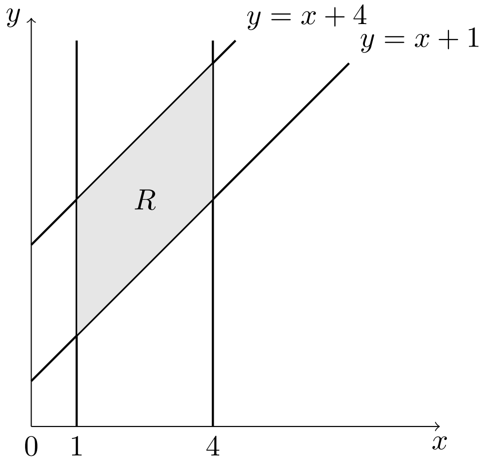
$$\int_1^4 \int_{x+1}^{x+4} \frac{1}{\sqrt{xy - x^2}}\, dy\, dx$$

Under the transformation T given by $x(u,v) = u$, $y(u,v) = u+v$, we see that the line $x = 1$, $x = 4$ are the image of the line $u = 1$ and $u = 4$. On the other hand,

$$\begin{aligned} y = x+1 &\iff u+v = u+1 \iff v = 1 \\ y = x+4 &\iff u+v = u+4 \iff v = 4 \end{aligned}$$

Therefore our region S is the region bounded by lines $u = 1$, $u = 4$, $v = 1$ and $v = 4$ (see figure below)





The change of variable formula is

$$\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

As $x(u,v) = u$, $y(u,v) = u+v$, $\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$

Then using the change of formula from above, we get

$$\begin{aligned} \int_1^4 \int_{x+1}^{x+4} \frac{1}{\sqrt{xy - x^2}}\, dy\, dx &= \int_1^4 \int_1^4 \frac{1}{\sqrt{u(u+v) - u^2}} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv \\ &= \int_1^4 \int_1^4 \frac{1}{\sqrt{uv}}\, du\, dv = \int_1^4 u^{-1/2}\, du \int_1^4 v^{-1/2}\, dv \\ &= 2u^{1/2}\Big|_1^4 \cdot 2v^{1/2}\Big|_1^4 = 4(2-1)(2-1) = 4 \end{aligned}$$

5. (15 points) Find the volume of the region E bounded by the paraboloids $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$

The volume of region E is given by

$$\text{Vol}(E) = \iiint_E 1\, dV$$

The paraboloids $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$ intersect at

$$x^2 + y^2 = 36 - 3x^2 - 3y^2 \implies 4x^2 + 4y^2 = 36 \implies x^2 + y^2 = 9$$

Then the region can be described in terms of cylindrical coordinates as

$$E = \{(r, \theta, z) \,|\, 0 \leq r \leq 3,\, 0 \leq \theta \leq 2\pi,\, r^2 \leq z \leq 36 - 3r^2\}$$

Therefore,

$$\begin{aligned}
\text{Vol}(E) = \iiint_E 1\, dV &= \int_0^{2\pi} \int_0^3 \int_{r^2}^{36-3r^2} r\, dz\, d\theta\, dr \\
&= \int_0^{2\pi} \int_0^3 rz \Big|_{z=r^2}^{z=36-3r^2} d\theta\, dr \\
&= \int_0^{2\pi} \int_0^3 r(36 - 3r^2 - r^2)\, d\theta\, dr \\
&= \int_0^{2\pi} d\theta \int_0^3 36r - 4r^3\, dr \\
&= 2\pi \left[18r^2 - r^4\right]_0^3 \\
&= 2\pi(162 - 81) \\
&= 162\pi
\end{aligned}$$

6. (15 points) Evaluate the integral

$$\iiint_E xy\,dV$$

where E is the region above the cone $z = \sqrt{x^2+y^2}$, below the sphere $x^2+y^2+z^2 = 16$ and in the **first** octant.

The cone $z = \sqrt{x^2+y^2}$ can be expressed in spherical coordinates by $\phi = \frac{\pi}{4}$, and the sphere $x^2+y^2+z^2 = 16$ can be expressed as $\rho = 4$.

Then the region E can be described in spherical coordinates as

$$E = \left\{(\rho,\theta,\phi)\,|\,0 \le \rho \le 4,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi \le \frac{\pi}{4}\right\}$$

Then,

$$\begin{aligned}
\iiint_E xy\,dV &= \int_0^4 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} (\rho^2 \sin\phi\cos\theta)(\rho\sin\phi\sin\theta)\rho\sin\phi\,d\phi\,d\theta\,d\rho \\
&= \int_0^4 \rho^4\,d\rho \int_0^{\pi/2} \cos\theta\sin\theta\,d\theta \int_0^{\pi/4} \sin^3\phi\,d\phi \\
&= \left[\frac{\rho^5}{5}\right]_{\rho=0}^{\rho=4} \left[\frac{\sin^2\theta}{2}\right]_{\theta=0}^{\theta=\pi/2} \int_0^{\pi/4} \sin\phi(1-\cos^2\phi)\,d\phi \\
&= \frac{4^5}{5} \cdot \left(\frac{1}{2} - 0\right) \int_0^{\pi/4} \sin\phi - \sin\phi\cos^2\phi\,d\phi \\
&= \frac{512}{5} \cdot \left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi/4} \\
&= \frac{512}{5} \cdot \left[-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{12} - \left(-1 + \frac{1}{3}\right)\right] \\
&= \frac{128}{15}(8 - 5\sqrt{2})
\end{aligned}$$

7. (15 points) (a) (11 points) Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$, where

$$f(u,v,w) = \sqrt{u^2+v^2+w^2}, \quad u = xe^y,\ v = ye^x,\ w = e^{xy}$$

$$\frac{\partial f}{\partial u} = \frac{u}{\sqrt{u^2+v^2+w^2}}; \quad \frac{\partial f}{\partial v} = \frac{v}{\sqrt{u^2+v^2+w^2}}; \quad \frac{\partial f}{\partial w} = \frac{w}{\sqrt{u^2+v^2+w^2}}$$

$$\frac{\partial u}{\partial x} = e^y; \quad \frac{\partial v}{\partial x} = ye^x; \quad \frac{\partial w}{\partial x} = ye^{xy}$$

$$\frac{\partial u}{\partial y} = xe^y; \quad \frac{\partial v}{\partial y} = e^x; \quad \frac{\partial w}{\partial y} = xe^{xy}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial v}{\partial x} = \frac{xe^y e^y + ye^x ye^x + e^{xy} ye^{xy}}{\sqrt{(xe^y)^2 + (ye^x)^2 + (e^{xy})^2}}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial v}{\partial y} = \frac{xe^y xe^y + ye^x e^x + e^{xy} xe^{xy}}{\sqrt{(xe^y)^2 + (ye^x)^2 + (e^{xy})^2}}$$

(b) (4 points) Recall that the equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is given by

$$\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

Use you answer in part (a) to find the equation of the tangent plane to the curve $z = f(x, y)$, when $(x_0, y_0) = (0, 2)$.

$$u(0,2) = 0,\ v(0,2) = 2,\ w(0,2) = 1$$

$$f_x(0,2) = \frac{6}{\sqrt{4+1}} = \frac{6}{\sqrt{5}};\ f_y(0,2) = \frac{2}{\sqrt{5}}$$
$$z_0 = f(0,2) = \sqrt{0^2 + 2^2 + 1^2} = \sqrt{5}$$

The the equation of the tangent plane is given by

$$z - \sqrt{5} = \frac{6}{\sqrt{5}}(x - 0) + \frac{2}{\sqrt{5}}(y - 2)$$

8. (5 points) A function f of two variables is called **continuous** at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

Determine if the following function

$$f(x, y) = \begin{cases} \dfrac{x^4 - 4y^2}{x^2 + 2y^2} & \text{if } (x, y) \neq (0, 0) \\ 3 & \text{if} (x, y) = (0, 0) \end{cases}$$

is continuous at $(0, 0)$.

First we need to compute $\lim\limits_{(x,y)\to(0,0)} f(x, y)$.

If we approach $(0, 0)$ along along the x-axis, we have $y = 0$. Then $f(x, 0) = \dfrac{x^4}{x^2} = x^2$ for all $x \neq 0$. Therefore $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along the x-axis.

If we approach $(0, 0)$ along along the y-axis, we have $x = 0$. Then $f(0, y) = \dfrac{-4y^2}{2y^2} = -2$ for all $y \neq 0$. Therefore $f(x, y) \to -2$ as $(x, y) \to (0, 0)$ along the y-axis.

Since f has two different limits along two different lines, $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

Since the limit does not exist, f is not continuous at $(0, 0)$.